

More on the Nambu-Poisson M5-brane Theory

*Scaling limit, background independence and
an all order solution to the Seiberg-Witten map*

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Abstract

We continue our investigation on the Nambu-Poisson description of M5-brane in a large constant C -field background (NP M5-brane theory) constructed in Refs.[1, 2]. In this paper, the low energy limit where the NP M5-brane theory is applicable is clarified. The background independence of the NP M5-brane theory is made manifest using the variables in the BLG model of multiple M2-branes. An all order solution to the Seiberg-Witten map is also constructed.

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1 Introduction

M2-branes and M5-branes are the fundamental building blocks of M-theory. As a way to understand the mysterious nature of M-theory, it is desirable to understand them as much as possible. However, for the time being our understanding of the M-branes are far less comprehensive than our understanding of D-branes in string theory, and comparatively the M5-brane is even less understood than the M2-brane.

The solitonic solution for M5-branes [3] in 11 dimensional supergravity was found even before the advent of M-theory [4, 5]. After that, people successfully constructed the equations of motion [6, 7] and then the action for a single M5-brane [8, 9, 10, 11]. The quantum aspects of a single M5-brane were also understood to some extent [12, 13]. However, while we understand the physics of multiple D-branes and non-commutative D-branes, the analogous knowledge about M5-branes is absent. In this paper, we make an effort to understand a related problem— the physics of M5-brane in C -field background.

Recently, a new worldvolume action describing a single M5-brane in a large constant C -field background was constructed [1, 2]. This action was obtained from the BLG model of multiple M2-branes [14, 15, 16], which has a gauge symmetry based on Lie 3-algebra. By choosing the Nambu-Poisson (NP) structure as the Lie 3-algebra and expanding around a certain background, one obtains the new M5-brane action. We will refer to this theory as *NP M5-brane theory* in the following. The construction is analogous to that of a $D(p+2)$ -brane in a constant B -field background from infinitely many Dp -branes, and in fact it can be uplifted to M-theory in certain cases through the relation between M-theory and type IIA superstring theory. Since extensive research has been made on D-branes in a constant B -field background (see e.g. [17] and references therein), the uplift to M-theory will give us good clues to understand the M5-brane worldvolume theory. Indeed, taking the analogy with D-branes in a constant B -field background as a guidance, in Refs.[1, 2] it was conjectured that the NP M5-brane theory is related to the conventional description of M5-brane [6, 7, 8, 9, 10, 11] in a constant C -field background through the so-called Seiberg-Witten map [18]. In the case of D-branes in a constant B -field background one has two descriptions of the same system, the one using ordinary coordinates and another with non-commutative coordinates. Seiberg-Witten map relates the ordinary description and the non-commutative description of D-branes in a constant B -field background. Therefore, this conjecture is a natural extension of such a D-brane system to an M5-brane in a constant C -field background. In Ref.[2], the Seiberg-Witten map for M5-brane was constructed up to the first order in a parameter which parametrizes the strength of the interaction through the NP bracket. We remind the reader that in the case of D-branes in a constant B -field background, non-commutative description was practically much more convenient than the ordinary description in the zero-slope limit [18], and the same will be true for the M5-brane in a constant C -field background. In this sense, we may say that the NP bracket description captures the structure of the M5-brane worldvolume theory in this background in a more essential way.

Several non-trivial supports for this conjecture have been given in the original papers as well as in subsequent works. In the original papers [1, 2], it was shown that the NP M5-brane theory has the same field contents with conventional M5-branes as well as the six-dimensional (2,0) supersymmetry. It was also shown that the double dimensional reduction of the NP M5-brane theory reduces to the Poisson description of D4-brane in a constant B -field background in a rather non-trivial way. This, through the M-theory – IIA string relation, provides an indirect support for the identification of NP M5-brane theory as a theory of M-theory five-brane. Furthermore, an argument within M-theory based on the central charge of the eleven-dimensional super-Poincare algebra (or “M-theory superalgebra”) in the BLG model was given in [19]. In [20], the BPS string solitons on the NP M5-brane worldvolume was constructed and compared with the corresponding object in the ordinary description [21, 22, 23] via the Seiberg-Witten map, and precise match was found up to the first order in the NP parameter for the scalar field configurations. The test was further extended to the comparison of defining BPS equations for string solitons between two descriptions in [24]. Notice that these are also direct tests of the conjecture without referring to the M-theory – IIA string relation. And it has been clarified how the self-dual relations, which is a salient feature of the M5-brane theory [6, 7, 8, 25, 26], are encoded in the NP M5-brane action [2, 27, 24]. With those evidences in hand, now the conjectured equivalence between two descriptions of M5-brane worldvolume theory in a constant C -field background has become very plausible. This in turn would provide a support for the validity of the BLG model as a description of M-theory branes.

In this paper, we will collect further evidences for the conjectured equivalence between the NP M5-brane theory and conventional M5-brane theory in a constant C -field background. In section 2, we recall some background materials which are useful for later sections. In section 3, we identify the low energy limit where the NP M5-brane theory is applicable. In section 4, the background independence of the NP M5-brane theory is made manifest by using the background independent variables in the BLG model. We also identify “open membrane metric” which governs the propagation of fields in the NP M5-brane worldvolume, and effective tension of the NP M5-brane. In section 5, we construct an all order solution to the Seiberg-Witten map. We end this paper with the discussions on future directions.

2 Preliminaries

In this section, we recall some background materials which are useful in later sections.

2.1 The BLG model of multiple M2-branes

The M5-brane action of Refs.[1, 2] was constructed from the Bagger-Lambert-Gustavsson model (BLG model) of multiple M2-branes [14, 15, 16]. This model has a novel type of gauge symmetry based on an algebraic structure called Lie 3-algebra [28]. For a linear space $\mathcal{V} =$

$\{\sum_{a=1}^{\dim \mathcal{V}} v_a T^a; v_a \in \mathbb{C}\}$, Lie 3-algebra structure is defined by a tri-linear map which is called 3-bracket $[\ast, \ast, \ast] : \mathcal{V}^{\otimes 3} \rightarrow \mathcal{V}$, satisfying the following properties:

1. Skew-symmetry:

$$[A_{\sigma(1)}, A_{\sigma(2)}, A_{\sigma(3)}] = (-1)^{|\sigma|} [A_1, A_2, A_3]. \quad (2.1)$$

2. Fundamental identity:

$$\begin{aligned} & [A_1, A_2, [B_1, B_2, B_3]] \\ = & [[A_1, A_2, B_1], B_2, B_3] + [B_1, [A_1, A_2, B_2], B_3] + [B_1, B_2, [A_1, A_2, B_3]]. \end{aligned} \quad (2.2)$$

A linear space endowed with a Lie 3-algebra structure will be called Lie 3-algebra. In terms of the basis T^a , Lie 3-algebra can be expressed in terms of the structure constants f^{abc}_d :

$$[T^a, T^b, T^c] = f^{abc}_d T^d. \quad (2.3)$$

We will be interested in Lie 3-algebra with inner product $\langle \ast, \ast \rangle : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathbb{C}$ (metric Lie 3-algebra):

$$\langle T^a, T^b \rangle = h^{ab}, \quad (2.4)$$

so that we can construct an action. We refer to h^{ab} as metric of the Lie 3-algebra. We require following invariance of the inner product which is needed for the gauge invariance of the BLG model:

$$\langle [T^a, T^b, T^c], T^d \rangle + \langle T^c, [T^a, T^b, T^d] \rangle = 0. \quad (2.5)$$

Together with the skew-symmetry property (2.1), the invariance of the metric (2.5) requires the indices of structure constants $f^{abcd} \equiv f^{abc}_e h^{ed}$ to be totally anti-symmetric:

$$f^{abcd} = \frac{1}{4!} f^{[abcd]}. \quad (2.6)$$

The action of the BLG model is given by

$$S = \int d^3x \mathcal{L}, \quad (2.7)$$

where the Lagrangian density \mathcal{L} is given by

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} \langle D^\mu \phi^I, D_\mu \phi^I \rangle + \frac{i}{2} \langle \bar{\Psi}, \Gamma^\mu D_\mu \Psi \rangle + \frac{i}{4} \langle \bar{\Psi}, \Gamma_{IJ} [\phi^I, \phi^J, \Psi] \rangle \\ & - V(\phi) + \mathcal{L}_{CS}. \end{aligned} \quad (2.8)$$

$\phi^I = \phi_a^I T^a$ ($I = 1, \dots, 8$) are scalar fields on the worldvolume which describe embedding of the M2-brane worldvolume in the transverse eight dimensions in the eleven-dimensional target

space-time. $\Psi = \Psi_a T^a$ are Majorana spinors on 1+2 dimensional worldvolume, but can be combined into a single Majorana spinor in eleven dimensions subject to the chirality condition $\Gamma\Psi = -\Psi$, $\Gamma \equiv \Gamma_{012}$. D_μ is the covariant derivative

$$(D_\mu \varphi(x))_a = \partial_\mu \varphi_a(x) - \tilde{A}_\mu{}^b{}_a(x) \varphi_b(x), \quad \tilde{A}_\mu{}^b{}_a \equiv A_{\mu cd} f^{cdb}{}_a, \quad (2.9)$$

where A_μ is the gauge field and φ collectively represents ϕ^I and Ψ . $V(\phi)$ is the potential

$$V(\phi) = \frac{1}{12} \langle [\phi^I, \phi^J, \phi^K], [\phi^I, \phi^J, \phi^K] \rangle. \quad (2.10)$$

The Chern-Simons term for the gauge potential is given by

$$\mathcal{L}_{CS} = \frac{1}{2} \varepsilon^{\mu\nu\lambda} \left(f^{abcd} A_{\mu ab} \partial_\nu A_{\lambda cd} + \frac{2}{3} f^{cda}{}_g f^{efgb} A_{\mu ab} A_{\nu cd} A_{\lambda ef} \right). \quad (2.11)$$

The action is invariant under the following gauge transformation:

$$\begin{aligned} \delta_\Lambda \phi_a^I &= \Lambda_{cd} [T^c, T^d, \phi^I]_a = \Lambda_{cd} f^{cde}{}_a \phi_e^I = \tilde{\Lambda}^e{}_a \phi_e^I, \\ \delta_\Lambda \Psi_a &= \Lambda_{cd} [T^c, T^d, \Psi]_a = \Lambda_{cd} f^{cde}{}_a \Psi_e = \tilde{\Lambda}^e{}_a \Psi_e, \\ \delta_\Lambda \tilde{A}_\mu{}^b{}_a &= \partial_\mu \tilde{\Lambda}_\mu{}^b{}_a - \tilde{\Lambda}^b{}_c \tilde{A}_\mu{}^c{}_a + \tilde{A}_\mu{}^b{}_c \tilde{\Lambda}^c{}_a, \quad \tilde{\Lambda}^b{}_a \equiv f^{cdb}{}_a \Lambda_{cd}. \end{aligned} \quad (2.12)$$

2.2 M-theory – type IIA superstring relation

M-theory and type IIA superstring theory are related by a circle compactification of M-theory. We will study an M5-brane in a constant C -field background, which is an uplift of a D4-brane in a constant B -field background. Although in the case of D-branes in a constant B -field background information was extracted from the worldsheet theory of open string, extracting information from the quantization of M2-brane worldvolume theory in a constant C -field background is more complicated (see [29, 30, 31, 32, 33] for some approaches from the M2-brane worldvolume). Instead, we will make use of the M-theory – IIA string relation to uplift the results in the D-branes in a constant B -field background. We briefly review the M-theory – IIA string relation in this subsection.

Let us consider a compactification of M-theory on a circle with the coordinate compactification radius R_{10}^{coord} (here we compactify the x^{10} direction). We take the coordinates where the ten-dimensional part of the background metric of M-theory and that of type IIA string theory are the same:

$$g_{\mu\nu}(M) = g_{\mu\nu}(IIA) \equiv g_{\mu\nu}, \quad \text{for } \mu, \nu = 0, \dots, 9. \quad (2.13)$$

The relation of the M-theory parameters and those in type IIA superstring theory are given as

$$R_{10}^{phys} = g_s \ell_s, \quad \ell_P = g_s^{1/3} \ell_s, \quad (2.14)$$

where R_{10}^{phys} is the physical compactification radius measured by the M-theory metric $g_{10,10(M)}$

$$(R_{10}^{phys})^2 \equiv g_{10,10(M)}(R_{10}^{coord})^2, \quad (2.15)$$

and $\ell_s \equiv (\alpha')^{1/2}$ and ℓ_P is the eleven-dimensional Planck scale (we follow the convention in Polchinski's text book [34]) which is related to the M-theory brane tensions as

$$T_{M2} = \frac{1}{(2\pi)^2 \ell_P^3}, \quad T_{M5} = \frac{1}{2\pi} (T_{M2})^2 = \frac{1}{(2\pi)^5 \ell_P^6}. \quad (2.16)$$

By the circle compactification of M-theory, M2-branes which wrap on the circle become fundamental strings, and those which do not wrap on the circle become D2-branes. Similarly, M5-branes which wrap on the circle become D4-branes, and those which do not wrap on the circle become NS5-branes. The M-theory – IIA string relation (2.14) correctly reproduces the tensions of D2-brane, D4-brane, fundamental string and NS5-brane which are given by

$$T_{Dp} = \frac{1}{(2\pi)^p g_s \ell_s^{p+1}}, \quad (2.17)$$

$$T_{F1} = \frac{1}{2\pi \alpha'}, \quad (2.18)$$

$$T_{NS5} = \frac{1}{(2\pi)^5 g_s^2 \ell_s^6}. \quad (2.19)$$

2.3 Open string theory in a constant B -field background

Open string theory on D-branes in a constant B -field background can be described by gauge theory on non-commutative space [35, 36, 37, 18]. Many interesting results have been obtained, such as Seiberg-Witten map [18], non-commutative instantons/solitons [38, 39, 40, 41, 42] and UV-IR mixing [43]. Some of these results should have corresponding uplift in M-theory via the M-theory – IIA string relation discussed in the previous section, which we would like to investigate. Let us briefly review some results in open strings in a constant B -field background.

In a constant B -field background, the propagation of open strings is governed by the so-called open string metric, and the effective coupling constant is also modified, which is often called open string coupling. Those are given as [18]

$$\begin{aligned} \left(\frac{1}{G + 2\pi\alpha'\Phi} + \frac{\theta}{2\pi\alpha'} \right)^{ij} &= \left(\frac{1}{g + 2\pi\alpha'B} \right)^{ij}, \\ G_s &= g_s \left(\frac{\det(G + 2\pi\alpha'\Phi)}{\det(g + 2\pi\alpha'B)} \right)^{1/2}, \end{aligned} \quad (2.20)$$

where G_{ij} is the open string metric and G_s is the open string coupling. Φ parametrizes a freedom in the description [18]. A natural choice for Φ is

$$\Phi = -B, \quad (2.21)$$

which leads to

$$G^{ij} = -\frac{1}{(2\pi\alpha')^2} \left(\frac{1}{B} g \frac{1}{B} \right)^{ij}, \quad (2.22)$$

$$G_{ij} = -(2\pi\alpha')^2 (Bg^{-1}B)_{ij}, \quad (2.23)$$

$$\theta^{ij} = \left(\frac{1}{B} \right)^{ij}, \quad (2.24)$$

$$G_s = g_s \det(2\pi\alpha' Bg^{-1})^{1/2}. \quad (2.25)$$

In general, even if we restrict ourselves to the massless sector, the low energy effective field theory on D-branes in such background still receives α' corrections and is described by an action like Nambu-Goto-Dirac-Born-Infeld type action on non-commutative space, or with further α' corrections. On the other hand, the non-commutative Yang-Mills theory (NCYM)¹ is obtained in a particular zero-slope limit [18]:

$$\begin{aligned} \alpha' &\sim \epsilon^{1/2} \rightarrow 0, \\ g_{ij} &\sim \epsilon \rightarrow 0. \end{aligned} \quad (2.26)$$

Notice that this limit with finite B leads to the finite open string metric. The Yang-Mills coupling on the Dp-brane is given by

$$\frac{1}{g_{YM}^2} = \frac{(\alpha')^{\frac{3-p}{2}}}{(2\pi)^{p-2} G_s} = \frac{(\alpha')^{\frac{3-p}{2}}}{(2\pi)^{p-2} g_s} \left(\frac{\det(g + 2\pi\alpha' B)}{\det G} \right)^{\frac{1}{2}}. \quad (2.27)$$

From (2.27) it follows that to obtain a finite Yang-Mills coupling in the zero-slope limit (2.26), we should scale g_s and G_s as

$$\begin{aligned} G_s &\sim \epsilon^{\frac{3-p}{4}}, \\ g_s &\sim \epsilon^{\frac{3-p+r}{4}}, \end{aligned} \quad (2.28)$$

where r is the rank of the background B -field.

3 The NP M5-brane theory limit from the zero-slope limit of type IIA string theory in a constant B -field background

We would like to study the situation where the NP M5-brane theory reduces to the Yang-Mills theory on a Poisson manifold as the D4-brane worldvolume theory upon double dimensional

¹By Yang-Mills theory we refer to the theory described by the action with the curvature square term $\text{tr} F^2$. We include the case where the gauge group is $U(1)$ for brevity, since on the non-commutative space the action for the case with $U(1)$ gauge group takes the similar form to that of the case with $U(N)$ gauge group due to the self-coupling of the gauge fields through the non-commutativity.

reduction. Here, the description on a Poisson manifold can be regarded either as a small non-commutativity approximation of the Moyal product description (explained in section 4), or another description of the D4-brane in a constant B -field background. In the former case, using the M-theory – type IIA superstring relation reviewed in the previous subsection, we can translate the scaling to the non-commutative Yang-Mills description in type IIA superstring discussed in subsection 2.3 to the scaling limit which leads to the NP M5-brane theory.² However, one should be aware of the difficulty in obtaining the non-commutative Yang-Mills description of D4-brane from the double dimensional reduction of the deformation of NP M5-brane theory [44]. On the other hand, when we take the latter interpretation, we will assume that the open string metric and open string coupling are the same both in the non-commutative description and the Poisson description of the D4-brane in a B -field background.

Now let us consider the double dimensional reduction of the NP M5-brane action. Here, we study the configuration where the worldvolume of the NP M5-brane extends in (012345)-directions, among which (012) were the worldvolume directions of the original multiple M2-branes. Unlike sec. 2.2, in this section we compactify the x^5 -direction instead of the x^{10} -direction. Then, the M-IIA relation (2.14) together with the zero-slope limit (2.26) enforce the following scaling of the parameters in M-theory:

$$\ell_P \sim \epsilon^{1/3}, \quad (3.1)$$

$$R_5^{phys} \sim \epsilon^{1/2}, \quad (3.2)$$

where R_5^{phys} is the physical compactification radius in the 5-th direction. Eq. (3.1) ensures the decoupling of the eleven-dimensional gravity (more discussions on this point later.) Eq. (3.2) means that if we fix (i.e., do not scale with ϵ) the coordinate compactification length, the g_{55} component of the metric scales as $g_{55} \sim \epsilon$ in the zero-slope limit (2.26). Notice that this behavior is the same as the scaling of g_{33} and g_{44} in (2.26). We take the coordinate compactification radius as R_5^{coord} . It is related to the physical compactification radius R_5^{phys} as follows:

$$(R_5^{phys})^2 = g_{55}(R_5^{coord})^2. \quad (3.3)$$

The C -field in M-theory is related to the B -field in IIA string theory as

$$C_{345}(2\pi R_5^{coord}) = B_{34}. \quad (3.4)$$

Summarizing, we can define the NP M5-brane theory limit by

$$\begin{aligned} \ell_P &\sim \epsilon^{1/3}, \\ g_{ij(M)} &\sim \epsilon, \\ C_{ijk} &\sim \epsilon^0 \quad (i, j = 3, 4, 5). \end{aligned} \quad (3.5)$$

²We assume that we are working in a particular choice of the freedom in the descriptions which might be there in the M5-brane theory, as in (2.20) in the case of D-branes in a constant B -field background.

Notice that although the scaling limit (3.5) was extracted from the scaling limit (2.26) in IIA string theory via the M-theory - IIA string relation, the limit itself can be taken without the compactification in the x^5 direction. In other words, the limit (3.5) can be studied totally within M-theory.

If we further tune the scaling in (3.5) so that the effective tension of M2-branes becomes finite, we arrive at the so-called OM-theory [45]. However, since we are interested in the field theory description of M5-brane, we will not consider the limit to the OM-theory.

The scaling of the C_{012} component of the background C -field is not independently chosen from the NP M5-brane limit (3.5), since it must obey the non-linear self-dual relations [6, 7]:

$$\frac{\sqrt{-\det g}}{6} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6} C^{\mu_4 \mu_5 \mu_6} = \frac{1+K}{2} g_{\mu_1 \mu} (\tilde{G}^{-1})^{\mu \nu} C_{\nu \mu_2 \mu_3}, \quad (\mu_1, \dots, \mu_6, \mu, \nu = 0, \dots, 5), \quad (3.6)$$

where $\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6}$ is a totally anti-symmetric tensor with $\epsilon_{012345} = 1$, and

$$K = \sqrt{1 + \frac{1}{24} (2\pi)^4 \ell_P^6 C^2}, \quad \tilde{G}_{\mu\nu} = \frac{1+K}{2K} \left(g_{\mu\nu} + \frac{1}{4} (2\pi)^4 \ell_P^6 C_{\mu\nu}^2 \right), \quad (3.7)$$

$$C^2 \equiv C_{\mu_1 \mu_2 \mu_3} C_{\nu_1 \nu_2 \nu_3} g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} g^{\mu_3 \nu_3}, \quad (C^2)_{\mu\nu} \equiv C_{\mu \mu_2 \mu_3} C_{\nu \nu_2 \nu_3} g^{\mu_2 \nu_2} g^{\mu_3 \nu_3}. \quad (3.8)$$

Using (3.6), one can check that $C_{012} \sim \epsilon^{-1}$ with the finite metric $g_{\mu\nu} = \eta_{\mu\nu}$ (for $\mu, \nu = 0, 1, 2$), using the non-linear self-dual relations for ordinary M5-brane. On the other hand, if we use the coordinates where $g_{ij} = \eta_{ij}$ ($i, j = 3, 4, 5$), the NP M5-brane limit (3.5) amounts to take $C_{345} \sim \epsilon^{-3/2}$ (note that $C_{\mu\nu\rho}$ is a tensor and the value of the components depend on the coordinate system). Comparing with this, one expects that C_{012} is not strong enough to induce finite interaction through the NP bracket in the (012)-directions in the scaling limit (3.5) (as long as we do not tune C_{012} to reach to the OM-theory). We give more explicit arguments in appendix A.

The NP M5-brane action is an analogue of the Poisson bracket Yang-Mills action on D4-brane, and indeed it reduces to it upon double dimensional reduction albeit rather non-trivially [2] (notice that the self-dual two-form gauge field on the M5-brane reduces to the one-form gauge field on D4-brane without the self-dual relations). The use of the Poisson bracket Yang-Mills action is justified in the particular scaling limit (2.26) from which we obtained the scaling limit (3.5). Therefore, the scaling limit (3.5) is also required to justify the use of the NP M5-brane action.

One would like to describe the M5-brane theory obtained through the scaling limit (3.5) by quantities which remain finite in this limit, analogous to the open string metric (2.23) and the non-commutative parameter (2.24) in the case of open string theory in a constant B -field background. This will be achieved in section 4.

4 Background independence of NP M5-brane theory and open membrane metric

In [46], it was shown that when we obtain NCYM as an expansion around a background in the matrix model, the background independence becomes manifest. Similar story holds when we construct NP M5-brane action from an expansion around a background in the BLG model.

At the time when uplift of open string theory in a constant B -field background to M-theory was studied, the “open membrane metric” as the M-theory analogue of open string metric [47, 48, 49, 50, 51, 52] has been proposed. We make an observation that the open membrane metric (in the scaling limit (3.5)) appears rather naturally in the kinetic term of the embedding coordinate fields in our construction. The effective tension of the NP M5-brane can also be read off from the kinetic term of the embedding coordinate fields.

4.1 Manifest background independence of NCYM on a D4-brane from D2-branes

Let us first recall the background independence of NCYM discussed in [18, 46]. The background independence here means that we hold closed string variables g_s and g_{ij} fixed when we vary the non-commutative parameter θ^{ij} .

For our purpose of comparing the NP M5-brane action with the NCYM action on D4-brane with the non-commutativity in (34)-directions, it is convenient to start from the action for multiple D2-branes. The potential term in the low energy effective action on D2-branes is given by

$$\frac{1}{(2\pi)^2 g_s \ell_s^3} \int d^3x \frac{1}{(2\pi\alpha')^2} \frac{1}{4} g_{II'} g_{JJ'} \text{tr}[X^I, X^J][X^{I'}, X^{J'}], \quad (I, J = 3, \dots, 9), \quad (4.1)$$

where X^I 's are Hermitian matrices with mass dimension $[X] = -1$. Here and throughout this paper, we will use $[A]$ to express the mass dimension of a quantity A .

Let us consider the background $X_{bg}^i = \hat{x}^i$ ($i = 3, 4$) satisfying

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \quad (4.2)$$

where θ^{ij} is an anti-symmetric constant tensor with mass dimension $[\theta^{ij}] = -2$. The algebra (4.2) can be realized by matrices with infinite size, which is interpreted as infinitely many D2-branes. We parametrize the fluctuation around the background (4.2) as

$$X^i = \hat{x}^i + \theta^{ij} \hat{A}_j(\hat{x}). \quad (4.3)$$

The mass dimension of \hat{A}_i is $[\hat{A}_i] = 1$, which is the standard mass dimension when the Yang-Mills coupling is an overall factor of the Yang-Mills action. To discuss the background independence, it is convenient to introduce variables C_i as

$$C_i \equiv B_{ij} X^j = B_{ij} \hat{x}^j + \hat{A}_i, \quad (4.4)$$

where $B_{ij} = (\theta^{-1})_{ij}$, presuming the relation (2.24).

The covariant derivatives in NCYM can be written using C_i as

$$D_i \hat{\varphi} = \partial_i \hat{\varphi} - i[\hat{A}_i, \hat{\varphi}] = -i[C_i, \hat{\varphi}]. \quad (4.5)$$

It follows that

$$-i[C_i, C_j] = \hat{F}_{ij} - B_{ij}. \quad (4.6)$$

The open string metric and open string coupling are given as in (2.23), (2.25):

$$G_{ij} = -(2\pi\alpha')^2 (Bg^{-1}B)_{ij}, \quad (4.7)$$

$$G_s = g_s \det(2\pi\alpha' Bg^{-1})^{1/2} = g_s \sqrt{\det G} \frac{|\text{Pf} \theta|}{2\pi\alpha'}. \quad (4.8)$$

On the other hand, the algebra of trace-class infinite size matrices can be isomorphically mapped to the algebra of square-integrable functions on \mathbb{R}^2 , with the product given by the so-called Moyal (star) product (see e.g. [53]). If we denote the matrices by $\hat{\cdot}$ on the symbols, the map is given as

$$\hat{f}(\hat{x}) = \int d^2k f(k) e^{ik\hat{x}} \leftrightarrow f(x) = \int d^2k f(k) e^{ikx}, \quad (4.9)$$

$$\hat{f}\hat{h} \leftrightarrow f(x) * h(x), \quad (4.10)$$

$$\text{tr} \leftrightarrow \int \frac{d^2x}{2\pi} \frac{1}{|\text{Pf} \theta|}, \quad (4.11)$$

where $f(x)$ and $h(x)$ are square integrable functions on \mathbb{R}^2 , and $*$ is the Moyal product:

$$f(x) * h(x) \equiv e^{\frac{i}{2}\theta^{ij} \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^j}} f(z) h(x) \Big|_{z=x}. \quad (4.12)$$

Notice that in the lowest order in the expansion in θ^{ij} , the anti-symmetrized Moyal products reduces to the Poisson bracket:

$$\begin{aligned} f(x) * h(x) - h(x) * f(x) &= i\theta^{ij} \frac{\partial}{\partial x^i} f(x) \frac{\partial}{\partial x^j} h(x) + \mathcal{O}(\theta^2) \\ &= i\{f(x), h(x)\}_{\text{Poisson}} + \mathcal{O}(\theta^2), \end{aligned} \quad (4.13)$$

where the Poisson bracket is given by

$$\{f(x), h(x)\}_{\text{Poisson}} = \theta^{ij} \frac{\partial}{\partial x^i} f(x) \frac{\partial}{\partial x^j} h(x). \quad (4.14)$$

Using (4.9)–(4.11), we obtain

$$\begin{aligned}
& \frac{1}{(2\pi)^3 g_s \ell_s^3} \int d^3 x \frac{1}{(2\pi\alpha')^2} \frac{1}{4} g_{II'} g_{JJ'} \text{tr}[X^I, X^J][X^{I'}, X^{J'}] \quad (I, J = 3, \dots, 9) \\
&= \frac{1}{(2\pi)^5 G_s \ell_s^5} \int d^5 x \sqrt{\det G} \left[-(2\pi\alpha')^2 \frac{1}{4} G^{ii'} G^{jj'} (\hat{F}_{ij} - B_{ij})(\hat{F}_{i'j'} - B_{i'j'}) \right. \\
&\quad \left. + \frac{1}{2} g_{II'} G^{ii'} D_i X^I D_{i'} X^{I'} + \frac{1}{(2\pi\alpha')^2} \frac{1}{4} g_{II'} g_{JJ'} [X^I, X^J][X^{I'}, X^{J'}] \right] \\
&= \frac{1}{g_{YM}^2} \int d^5 x \sqrt{\det G} \left[-\frac{1}{4} G^{ii'} G^{jj'} (\hat{F}_{ij} - B_{ij})(\hat{F}_{i'j'} - B_{i'j'}) \right. \\
&\quad \left. + \frac{1}{2} g_{II'} G^{ii'} D_i \phi^I D_{i'} \phi^{I'} + \frac{1}{4} g_{II'} g_{JJ'} [\phi^I, \phi^J][\phi^{I'}, \phi^{J'}] \right], \\
&\quad (i, j = 3, 4; I, J = 5, \dots, 9),
\end{aligned} \tag{4.15}$$

where

$$\frac{1}{g_{YM}^2} \equiv \frac{(2\pi\alpha')^2}{(2\pi)^5 G_s \ell_s^5} = \frac{1}{(2\pi)^3 G_s \ell_s}, \tag{4.16}$$

and

$$\phi^I \equiv \frac{1}{2\pi\alpha'} X^I. \tag{4.17}$$

In the above, with a slight abuse of notation, we have identified matrices and functions on \mathbb{R}^2 through the map (4.9) and used the same symbols. (For example, \hat{F}_{ij} above should be read as function on \mathbb{R}^2 which is mapped from the matrix defined in (4.6) with the same symbol through the map (4.9)). As mentioned before, the background independence here means we hold the closed string variables g_s and g_{ij} fixed, and the change in the non-commutative parameter θ^{ij} arises only from the change of the background (4.2). In the first line of (4.15), the background independence is manifest since the change in the non-commutative parameter θ^{ij} is totally due to the choice of the background in (4.2) and the closed string metric and closed string coupling are fixed. Notice that not only on the left hand side but also on the right hand side of (4.11) the background independence of the measure is also clear, since the rescaling of the non-commutative parameter can be generated by the rescaling of coordinates x^i , which cancel with each other in the measure (4.11).

To discuss background independence without taking the zero-slope limit, we can consider a more general action, see [46]. On the other hand, we are interested in the zero-slope limit where the closed string metric g_{ij} is scaled as in (2.26).

4.2 Manifest background independence of NP M5-brane action from the BLG model

We would like to proceed in a parallel way when constructing the NP M5-brane action from the BLG model. To extract the essential point, we will focus on the potential term of the BLG model (2.10):

$$\int d^3x V(\phi) = \int d^3x \frac{1}{12} g_{II'} g_{JJ'} g_{KK'} \langle [\phi^I, \phi^J, \phi^K], [\phi^{I'}, \phi^{J'}, \phi^{K'}] \rangle, \quad (4.18)$$

where ϕ^I is a canonically normalized scalar field in three dimensions, i.e. $[\phi^I] = 1/2$. We have also introduced the target space metric g_{IJ} in (4.18) in order to take into account the scaling limit (3.5). We will refer to the target space metric g_{IJ} as “closed membrane metric,” taking an analogy with the closed string metric in the case of open string theory in a constant B -field background explained in subsection 2.3.

To obtain the target space interpretation, we define

$$X^I \equiv \phi^I ((2\pi)^{2/3} \ell_P)^{3/2}. \quad (4.19)$$

Then, X^I has a dimension of length, $[X^I] = -1$. The potential term (4.18) takes the form

$$\int d^3x V(X) = \frac{1}{(2\pi)^2 \ell_P^3} \int d^3x \frac{1}{(2\pi)^4 \ell_P^6} \frac{1}{12} g_{II'} g_{JJ'} g_{KK'} \langle [X^I, X^J, X^K], [X^{I'}, X^{J'}, X^{K'}] \rangle, \quad (I, J = 3, \dots, 10). \quad (4.20)$$

Next, we choose the Nambu-Poisson structure on \mathbb{R}^3 as the Lie 3-algebra structure of the BLG model (see subsection 2.1) [1, 2]. The Lie 3-bracket is given by the NP bracket

$$[A, B, C] = \{A, B, C\} = \theta^{ijk} \frac{\partial}{\partial x^i} A(x) \frac{\partial}{\partial x^j} B(x) \frac{\partial}{\partial x^k} C(x), \quad (i, j = 3, 4, 5). \quad (4.21)$$

Here, we choose θ^{ijk} to be a constant totally anti-symmetric tensor. The mass dimension of the Nambu-Poisson tensor θ^{ijk} is $[\theta^{ijk}] = -3$. Notice that the NP bracket is a natural generalization of the Poisson bracket (4.14).

The elements of the Lie 3-algebra are given by square-integrable functions on \mathbb{R}^3 .³ The inner product of the Lie 3-algebra is given by

$$\langle A, B \rangle = \int \frac{d^3x}{2\pi} \frac{1}{|\theta^{345}|} A(x) B(x). \quad (4.22)$$

The normalization of the inner product (4.22) is chosen in a parallel way to that in (4.11) in the case of matrix model (the choice of the 2π factor is for convenience in the comparison with NCYM upon double dimensional reduction). This choice ensures the background independence.

³The background needs not be square-integrable, and indeed the M5-brane background (4.23) which we will discuss shortly is an example of such background. See [19] for further discussions on this point, and see [54] for an alternative description of such background by introducing non-positive definite metric of the Lie 3-algebra.

The M5-brane extending in (012345)-direction is obtained by expanding the multiple M2-brane action extending in (012)-directions around the background

$$X_{bg}^i = x^i, \quad (i = 3, 4, 5). \quad (4.23)$$

We parametrize the fluctuation around the background as

$$X^i = x^i + \hat{b}^i(x) = x^i + \frac{1}{2}\theta^{ijk}\hat{b}_{jk}(x). \quad (4.24)$$

The mass dimension of the field \hat{b}_{ij} is $[\hat{b}_{ij}] = 2$. The field strength of the two-form gauge potential in (345)-directions is defined as

$$\hat{\mathcal{H}}_{ijk} = C_{ijk} \left(\frac{1}{6} C_{lmn} \left(\{X^\ell, X^m, X^n\} - \theta^{\ell mn} \right) \right), \quad (4.25)$$

where we have defined the totally anti-symmetric tensor C_{ijk} by

$$-C_{ijk}\theta^{k\ell m} = \delta_i^\ell \delta_j^m - \delta_i^m \delta_j^\ell. \quad (4.26)$$

We postulate that the tensor C_{ijk} is a component of the background C -field. We also postulate that the “(inverse) open membrane metric” is given by

$$G_{OM}^{ii'} = \frac{1}{2} \frac{\theta^{ijk}}{(2\pi)^2 \ell_P^3} \frac{\theta^{i'j'k'}}{(2\pi)^2 \ell_P^3} g_{jj'} g_{kk'} \quad (i, j, k = 3, 4, 5). \quad (4.27)$$

Not like in the case of string theory in a constant B -field background where we can read off the open string metric and the non-commutative parameter from the worldsheet two-point function, here we do not have a derivation of our postulates from the M2-brane worldvolume theory. We will show that our postulates are consistent with the open string metric and the open string coupling after the double dimensional reduction of the NP M5-brane action. This reasoning was basically the same as the one used in [50], though our study will be restricted to the scaling limit (3.5).

We define the covariant derivatives in (345)-directions as

$$D_i \hat{\varphi} \equiv -\frac{1}{2} C_{ijk} \{X^j, X^k, \hat{\varphi}\} \quad (i, j = 3, 4, 5). \quad (4.28)$$

Now the potential term (4.20) is rewritten as

$$\begin{aligned} & \int d^3x V(X) \\ = & T_6 \int d^6x \sqrt{\det G_{OM}} \left[\frac{1}{T_6} \frac{1}{2\pi} \frac{1}{12} G_{OM}^{ii'} G_{OM}^{jj'} G_{OM}^{kk'} (\hat{\mathcal{H}}_{ijk} - C_{ijk}) (\hat{\mathcal{H}}_{i'j'k'} - C_{i'j'k'}) \right. \\ & + \frac{1}{2} g_{II'} G_{OM}^{ii'} D_i X^I D_{i'} X^{I'} + \frac{1}{(2\pi)^4 \ell_P^6} \frac{1}{4} g_{ii'} g_{JJ'} g_{KK'} \langle [X^i, X^J, X^K], [X^{i'}, X^{J'}, X^{K'}] \rangle \\ & \left. + \frac{1}{(2\pi)^4 \ell_P^6} \frac{1}{12} g_{II'} g_{JJ'} g_{KK'} \langle [X^I, X^J, X^K], [X^{I'}, X^{J'}, X^{K'}] \rangle \right] \\ & (i, j = 3, 4, 5; I, J = 6, \dots, 10), \end{aligned} \quad (4.29)$$

where

$$T_6 \equiv \frac{T_{M2}}{2\pi|\theta^{345}|\sqrt{\det G_{OM}}} \quad (4.30)$$

is the effective tension of the NP M5-brane which is read off from the kinetic term for X^I . Notice that the effective tension of the NP M5-brane is much smaller than the fundamental M5-brane tension T_{M5} when $\theta \ll \ell_P^3$. Thus the back reaction to the closed membrane metric is negligible in the limit $\ell_P \rightarrow 0$. In the above, by $\det G_{OM}$ we mean the determinant of $(G_{OM})_{ii'}$ which is the inverse of $G_{OM}^{ii'}$ defined in (4.27). The rewriting of the gauge field kinetic term is not as obvious as in the case of non-commutative Yang-Mills theory on D4-brane from infinitely many D2-branes. Going back to the original variable ϕ^I in (4.19) for $I = 6, \dots, 10$, all the terms in (4.29) are also finite in the limit 3.5).

Now, we would like to examine the double dimensional reduction of the NP M5-brane action. We compactify the x^5 direction with a coordinate compactification radius R_5^{coord} as in subsection 2.2. R_5^{coord} is related to the physical compactification radius R_5^{phys} as

$$(R_5^{phys})^2 = g_{55}(R_5^{coord})^2. \quad (4.31)$$

The background C -field is related to the background B -field through the double dimensional reduction:

$$C_{345}(2\pi R_5^{coord}) = B_{34}. \quad (4.32)$$

Then, the non-commutative parameter θ^{34} and the Nambu-Poisson tensor θ^{345} are related as

$$\theta^{345} = \theta^{34}(2\pi R_5^{coord}), \quad (4.33)$$

by the postulate (4.26) with θ^{34} in (2.24). Actually, our postulate (4.26) was made so that it is consistent with θ^{34} in (2.24). Notice that (2.24) follows from our choice $\Phi = -B$ (2.21) in the freedom in the description (2.20). Thus our postulate for the Nambu-Poisson tensor (4.26) leads to the choice $\Phi = -B$ upon the circle compactification of M-theory. The relation (4.33) also follows from the double dimensional reduction of the NP M5-brane action. This is quite expected since the multiple M2-brane action should reduce to multiple D2-brane action upon circle compactification of M-theory, and the multiple D2-brane action naturally leads to the non-commutative D4-brane action with the choice (2.21) as we have seen in the previous subsection 4.1. See appendix B for the explicit calculations.

Let us examine the dimensional reduction of the (inverse) open membrane metric (4.27). We first examine the metric in (34)-directions. Using the relations (4.31)-(4.33) as well as the M-theory – IIA relation discussed in subsection 2.2, we obtain

$$G_{OM}^{ij} = G^{ij} \quad (i, j = 3, 4), \quad (4.34)$$

where G^{ij} is the inverse open string metric given in (2.22). Notice that (2.22) also follows from the choice (2.21). Thus our postulate for the (inverse) open membrane metric (4.27) also leads to the choice $\Phi = -B$ (2.21) upon the circle compactification of M-theory.

Regarding the G_{OM}^{55} component of the (inverse) open membrane metric, interestingly we have an analogue of M-IIA relation (2.14) for the open string/membrane variables [45]:

$$\sqrt{\frac{1}{G_{OM}^{55}}(R_5^{coord})^2} = G_s \ell_s. \quad (4.35)$$

4.3 Relation to the notation in Ref.[2]

In Ref.[2], an explicit parametrization of the Nambu-Poisson tensor was used. While it is convenient for actual calculations, when using this parametrization one should keep in mind that it is an expression in a particular coordinate system. In this paper, we keep track of the tensor structures so that we can keep manifest covariance, in particular, under the rescaling of the coordinates. This covariant tensor notation is useful when discussing ambiguities in the Seiberg-Witten map, as we will see in section 5.

In the following, we clarify the relation of the notation in our paper and that in Ref.[2].

From (4.23) we have

$$\{X_{bg}^3, X_{bg}^4, X_{bg}^5\} = \theta^{345}. \quad (4.36)$$

In Ref.[2], the 3-bracket was given as

$$[A, B, C] = g^2 \ell^3 \epsilon^{ijk} \frac{\partial}{\partial y^i} A(y) \frac{\partial}{\partial y^j} B(y) \frac{\partial}{\partial y^k} C(y), \quad (4.37)$$

where

$$x^i = \frac{y^i}{g}. \quad (4.38)$$

In the above, we have introduced a length scale ℓ so that the mass dimension of the 3-bracket is zero. The convention in Ref.[2] can be regarded as setting $\ell = 1$. From (4.37) and (4.38) we obtain

$$[X_{bg}^3, X_{bg}^4, X_{bg}^5] = \frac{\ell^3}{g}. \quad (4.39)$$

This should be compared with the 3-bracket in our convention (4.21). We obtain the relation

$$\theta^{345} = \frac{\ell^3}{g}. \quad (4.40)$$

Note that the expression in (4.40) is the component of the Nambu-Poisson tensor in the x coordinates.

The inner product in this paper was given as (4.22):

$$\int \frac{d^3x}{2\pi} \frac{1}{|\theta^{345}|}. \quad (4.41)$$

In terms of y coordinates, (4.41) becomes

$$\int \frac{d^3y}{2\pi} \frac{1}{g^3 |\theta^{345}|} = \int \frac{d^3y}{2\pi} \frac{1}{g^2 \ell^3}. \quad (4.42)$$

This is basically the same with eq.(3.5) of Ref.[2] after setting $\ell = 1$, up to the convention for the 2π factor.

In Ref.[2], the metric on the M5-brane in (345)-directions was given by the Kronecker delta. This can be achieved by first taking the closed membrane metric as

$$g_{ij} = (2\pi)^2 \frac{\ell_P^3}{\ell^3} \delta_{ij} \quad (i, j = 3, 4, 5). \quad (4.43)$$

Note that (4.43) follows the scaling (3.5). With this choice of closed string metric, the (inverse) open membrane metric becomes

$$G_{OM}^{ij} = \frac{1}{g^2} \delta^{ij}. \quad (4.44)$$

Then, via the coordinate transformation from x^i to y^i as in (4.38), the (inverse) open membrane metric becomes the Kronecker delta in y -coordinates:

$$G_{OMy}^{ij} = \delta^{ij}, \quad (4.45)$$

where we used the subscript y to indicate that (4.45) is the component expression in y coordinates. Since in y -coordinates the metric is kept fixed, it is a convenient coordinate system for measuring the physical strength of the interaction through the NP bracket.

In Ref.[2], the fields X^i ($i = 3, 4, 5$) were parametrized as

$$X^i = \frac{y^i}{g} + \hat{b}^{(g)i}(y) = \frac{y^i}{g} + \frac{1}{2} \epsilon^{ijk} \hat{b}_{jk}^{(g)}(y), \quad (4.46)$$

where we put superscript (g) to the corresponding variables in the notation of Ref.[2]. On the other hand, in this paper they were parametrized as

$$X^i = x^i + \hat{b}^i(x) = x^i + \frac{1}{2} \theta^{ijk} \hat{b}_{jk}(x). \quad (4.47)$$

To compare (4.47) with (4.46), we should first make coordinate transformation from x to y related by (4.38):

$$X^i = \frac{1}{g} (y^i + \hat{b}^i(y)) = \frac{1}{g} (y^i + \frac{1}{2} \theta_y^{ijk} \hat{b}_{jk}(y)), \quad (4.48)$$

where θ_y^{ijk} is the component of the Nambu-Poisson tensor in y coordinates, which from (4.40) is given by

$$\theta_y^{345} = g^3 \theta^{345} = g^2 \ell^3. \quad (4.49)$$

Note that as contravariant and covariant tensor fields, \hat{b}^i and \hat{b}_{ij} change under change of coordinates. Our notation is that we use $\hat{b}^i(x)$ and $\hat{b}^i(y)$ to denote the vector field \hat{b}^i in the x and y coordinate systems, respectively. As a result,

$$\hat{b}^i(y) = g \hat{b}^i(x), \quad \hat{b}_{ij}(y) = g^{-2} \hat{b}_{ij}(x). \quad (4.50)$$

Thus we obtain the relation between current convention and the convention in Ref.[2]:

$$\begin{aligned} \hat{b}^{(g)i}(y) &= \frac{1}{g} \hat{b}^i(y), \\ \hat{b}_{ij}^{(g)}(y) &= g \ell^3 \hat{b}_{ij}(y). \end{aligned} \quad (4.51)$$

In the BLG model, the covariant derivative was given in (2.9)

$$(D_\mu \varphi)_a = \partial_\mu \varphi_a - f^{bcd}{}_a A_{\mu bc} \varphi_d. \quad (4.52)$$

This can be rewritten as

$$D_\mu \varphi = \partial_\mu \varphi - A_{\mu bc} [T^b, T^c, \varphi]. \quad (4.53)$$

When we defined the Lie 3-algebra through the Nambu-Poisson structure on \mathbb{R}^3 , the elements T^a of the algebra were given by square-integrable functions on \mathbb{R}^3 .

We define the components of the two-form field $\hat{b}_{\mu i}$ by

$$\hat{b}_{\mu i} \equiv A_{\mu bc} T^b \partial_i T^c \quad (\mu = 0, 1, 2; i = 3, 4, 5). \quad (4.54)$$

Then, the covariant derivatives in (012)-directions in the NP M5-brane theory can be written as

$$D_\mu \hat{\varphi} = \partial_\mu \hat{\varphi} - \theta^{ijk} \frac{\partial}{\partial x^i} \hat{b}_{\mu j}(x) \frac{\partial}{\partial x^k} \hat{\varphi}. \quad (4.55)$$

This should be compared with the convention of Ref.[2]:

$$D_\mu \hat{\varphi}^{(g)} = \partial_\mu \hat{\varphi}^{(g)} - g \epsilon^{ijk} \frac{\partial}{\partial y^i} \hat{b}_{\mu j}^{(g)}(y) \frac{\partial}{\partial y^k} \hat{\varphi}^{(g)}. \quad (4.56)$$

Comparing them in the y coordinates, we obtain the relation

$$\hat{b}_{\mu i}^{(g)}(y) = g \ell^3 \hat{b}_{\mu i}(y). \quad (4.57)$$

5 An all order solution to the Seiberg-Witten map

5.1 Seiberg-Witten map

Seiberg-Witten map is a map between ordinary description and non-commutative description of a gauge theory, determined by the requirement that the gauge transformation for the non-commutative description is induced by the gauge transformation in the ordinary description.⁴ In the original paper by Seiberg and Witten [18], the map was explained by identifying the two descriptions as two different regularizations on the open string worldsheet theory in a constant B -field background. The difference in the regularization should not lead to different space-time S-matrices, and therefore fields in two descriptions should be related by field redefinitions. Later on, the Seiberg-Witten map between ordinary description and Poisson bracket description was explained as different gauge fixings of the reparametrization invariance on the D-brane world-volume [56, 57, 58] (see [59, 60, 61] for a related approach in constructing Seiberg-Witten map between ordinary description and Moyal-product description). We can follow a similar approach for constructing the Seiberg-Witten map between ordinary description and NP description of M5-brane in a constant C -field background.

In the case of M5-brane in a constant C -field background, Seiberg-Witten map is a solution to the condition: “Gauge transformations in the Nambu description is compatible with gauge transformations in the ordinary description”:

$$\hat{\delta}_\Lambda \hat{\Phi}(\Phi) = \hat{\Phi}(\Phi + \delta_\Lambda \Phi) - \hat{\Phi}(\Phi), \quad (5.1)$$

where $\hat{\Phi}(\Phi)$ collectively represents fields in the NP bracket (ordinary) description of M5-brane. The gauge transformation laws in the NP M5-brane theory were derived in [2], which together with those in the ordinary description of M5-brane we summarize in our notation below. The gauge transformation laws of b_{ij} and \hat{b}_{ij} are given by

$$\begin{aligned} \delta_\Lambda b_{ij} &= \partial_i \Lambda_j - \partial_j \Lambda_i, \\ \delta_\Lambda \hat{b}_{ij} &= \partial_i \hat{\Lambda}_j - \partial_j \hat{\Lambda}_i + \theta^{\ell mn} \partial_\ell \hat{\Lambda}_m \partial_n \hat{b}_{ij}. \end{aligned} \quad (5.2)$$

In terms of $b^i \equiv \frac{1}{2} \theta^{ijk} b_{jk}$ and $\hat{b}^i \equiv \frac{1}{2} \theta^{ijk} \hat{b}_{jk}$, these can be rewritten as

$$\delta_\Lambda b^i = \kappa^i, \quad \delta_\Lambda \hat{b}^i = \hat{\kappa}^i + \hat{\kappa}^j \partial_j \hat{b}^i, \quad (5.3)$$

where

$$\kappa^i \equiv \theta^{ijk} \partial_j \Lambda_k, \quad \hat{\kappa}^i \equiv \theta^{ijk} \partial_j \hat{\Lambda}_k. \quad (5.4)$$

From (5.4), the gauge transformation parameters κ^i and $\hat{\kappa}^i$ satisfy the divergenceless condition

$$\partial_i \kappa^i = 0, \quad \partial_i \hat{\kappa}^i = 0. \quad (5.5)$$

⁴Essentially the same problem had been considered in the study of fermions in the lowest Landau level [55].

Note that \hat{b}^i and $\hat{\kappa}^i$ are one order higher in the expansion in θ compared with \hat{b}_{ij} and $\hat{\Lambda}_i$, respectively. The gauge transformation laws of $b_{\mu i}$ and $\hat{b}_{\mu i}$ are given by

$$\begin{aligned}\delta_\Lambda b_{\mu i} &= \partial_\mu \Lambda_i - \partial_i \Lambda_\mu, \\ \delta_{\hat{\Lambda}} \hat{b}_{\mu i} &= \partial_\mu \hat{\Lambda}_i - \partial_i \hat{\Lambda}_\mu + \hat{\kappa}^j \partial_j \hat{b}_{\mu i} + \partial_i \hat{\kappa}^j \hat{b}_{\mu j}.\end{aligned}\tag{5.6}$$

The gauge transformation laws for φ and $\hat{\varphi}$ are given by

$$\delta_\Lambda \varphi = 0, \quad \delta_{\hat{\Lambda}} \hat{\varphi} = \hat{\kappa}^j \partial_j \hat{\varphi}.\tag{5.7}$$

The Seiberg-Witten map to the first order was obtained in [2] as

$$\hat{b}^i = b^i + \frac{1}{2} b^j \partial_j b^i + \frac{1}{2} b^i \partial_j b^j + \mathcal{O}(\theta^3),\tag{5.8}$$

$$\begin{aligned}\hat{B}_\mu{}^i &= B_\mu{}^i + b^j \partial_j B_\mu{}^i - \frac{1}{2} b^j \partial_\mu \partial_j b^i + \frac{1}{2} b^i \partial_\mu \partial_j b^j + \partial_j b^i B_\mu{}^j \\ &\quad - \frac{1}{2} \partial_j b^j \partial_\mu b^i + \frac{1}{2} \partial_j b^i \partial_\mu b^j + \mathcal{O}(\theta^3),\end{aligned}\tag{5.9}$$

$$\hat{\kappa}^i = \kappa^i + \frac{1}{2} b^j \partial_j \kappa^i + \frac{1}{2} \partial_j b^j \kappa^i - \frac{1}{2} \partial_j b^i \kappa^j + \mathcal{O}(\theta^3),\tag{5.10}$$

$$\hat{\varphi} = \varphi + \kappa^i \partial_i \varphi + \mathcal{O}(\theta^2).\tag{5.11}$$

Here we have used $\hat{B}_\mu{}^i$ (and $B_\mu{}^i$) defined below in place of $\hat{b}_{\mu i}$ (and $b_{\mu i}$) to express the Seiberg-Witten map. As mentioned earlier, in the expansion in θ we regard b_{ij} , $b_{\mu i}$, φ and Λ_i as $\mathcal{O}(1)$ variables, and thus count b^i , $B_\mu{}^i$ and κ^i as $\mathcal{O}(\theta)$. Thus, above expansion corresponds to the first order in θ expansion in terms of b_{ij} , $b_{\mu i}$, φ and Λ_i .

In the covariant derivatives and the action, $\hat{b}_{\mu i}$ always appears in the form

$$\hat{B}_\mu{}^i \equiv \theta^{ijk} \partial_j \hat{b}_{\mu k},\tag{5.12}$$

and thus we can avoid the explicit use of $\hat{b}_{\mu i}$ by using $\hat{B}_\mu{}^i$ with the constraint

$$\partial_i \hat{B}_\mu{}^i = 0,\tag{5.13}$$

which guarantees the existence of $\hat{b}_{\mu i}$ locally, and similarly for the variables without hats. For various purposes, $\hat{B}_\mu{}^i$ is more convenient to use. The gauge transformation laws of $\hat{B}_\mu{}^i$ and $B_\mu{}^i$ are given by

$$\delta_\Lambda B_\mu{}^i = \partial_\mu \Lambda^i,\tag{5.14}$$

$$\delta_{\hat{\Lambda}} \hat{B}_\mu{}^i = \partial_\mu \hat{\kappa}^i + \hat{\kappa}^j \partial_j \hat{B}_\mu{}^i - \partial_j \hat{\kappa}^i \hat{B}_\mu{}^j.\tag{5.15}$$

It turns out that it is easier to write down the Seiberg-Witten map for $\hat{B}_\mu{}^i$ than the one directly relating $\hat{b}_{\mu i}$ to $b_{\mu i}$.

To obtain an all order solution to the Seiberg-Witten map, we follow the approach of [60, 61] which was applied to the Poisson bracket $U(1)$ gauge theory. We will generalize their construction to the NP M5-brane theory. However in the case of NP M5-brane theory, there is the new ingredient that (012)-directions and (345)-directions are related through the self-dual relations for the two-form gauge field.

5.2 Nambu-Poisson manifold

To construct an all order solution to the Seiberg-Witten map, we need to extend our consideration to include general Nambu-Poisson manifold [62]. A Nambu-Poisson manifold is defined through a Nambu-Poisson bracket which is tri-linear and totally skew-symmetric in its entries:

$$\{A, B, C\} = \theta^{ijk}(x) \frac{\partial}{\partial x^i} A(x) \frac{\partial}{\partial x^j} B(x) \frac{\partial}{\partial x^k} C(x), \quad (i, j = 1, 2, 3), \quad (5.16)$$

and which satisfies the *fundamental identity*:

$$\{A, B, \{C, D, E\}\} = \{\{A, B, C\}D, E\} + \{C, \{A, B, D\}, E\} + \{C, D, \{A, B, E\}\}. \quad (5.17)$$

The fundamental identity (5.17) puts strong constraints on the totally anti-symmetric Nambu-Poisson tensor $\theta^{ijk}(x)$. For example, the following identities hold (see e.g. [63]):

$$\theta^{ib_2b_3}\theta^{ja_2b_1} + \theta^{b_1ib_3}\theta^{ja_2b_2} + \theta^{b_1b_2i}\theta^{ja_2b_3} + (i \leftrightarrow j) = 0, \quad (5.18)$$

$$\theta^{a_1a_2i}\partial_i\theta^{b_1b_2b_3} = \theta^{ib_2b_3}\partial_i\theta^{a_1a_2b_1} + \theta^{b_1ib_3}\partial_i\theta^{a_1a_2b_2} + \theta^{b_1b_2i}\partial_i\theta^{a_1a_2b_3}. \quad (5.19)$$

For a given Nambu-Poisson tensor θ^{ijk} , what we would like to have is a coordinate transformation $\rho(x^i)$:

$$\rho^*\theta^{ijk} = \theta^{ijk}, \quad (5.20)$$

i.e. a coordinate transformation that maps back the Nambu-Poisson bracket defined by θ^{ijk} to the original Nambu-Poisson bracket (5.16) defined by θ^{ijk} ,

$$\rho^*\{A, B, C\}' = \{\rho^*A, \rho^*B, \rho^*C\}, \quad (5.21)$$

where $\{*, *, *\}'$ is the Nambu-Poisson bracket defined by the Nambu-Poisson tensor θ^{ijk} . One can construct such a map by using a flow parametrized by t :

$$\partial_t\{A, B, C\} + \chi(t)\{A, B, C\} - \{\chi(t)A, B, C\} - \{A, \chi(t)B, C\} - \{A, B, \chi(t)C\} = 0, \quad (5.22)$$

where

$$\chi \equiv \frac{1}{2}\theta^{ijk}(t)b_{ij}\partial_k, \quad (5.23)$$

and

$$\theta^{ijk}(t=0) = \theta^{ijk}, \quad \theta^{ijk}(t=1) = \theta^{ijk}. \quad (5.24)$$

The tensor b_{ij} in (5.23) is related to θ^{ijk} as

$$-(C+H)_{ijk}\theta^{k\ell m} = \delta_i^\ell\delta_j^m - \delta_i^m\delta_j^\ell, \quad (5.25)$$

where

$$H_{ijk} \equiv \partial_i b_{jk} + \partial_j b_{ki} + \partial_k b_{ij}. \quad (5.26)$$

Using the fundamental identity (5.17), the flow equation (5.22) can be written as a first order differential equation for $\theta^{ijk}(t)$:

$$\begin{aligned} \partial_t \theta^{ijk}(t) &= \frac{1}{6} \left(\theta^{a_1 i j}(t) \theta^{a_2 a_3 k}(t) + \theta^{a_1 j k}(t) \theta^{a_2 a_3 i}(t) + \theta^{a_1 k i}(t) \theta^{a_2 a_3 j}(t) \right) H_{a_1 a_2 a_3} \\ &= \frac{1}{6} \theta^{ijk}(t) \theta^{a_1 a_2 a_3}(t) H_{a_1 a_2 a_3}. \end{aligned} \quad (5.27)$$

with the initial condition at $t = 0$ as in (5.24). The explicit solution for (5.27) with the initial condition mentioned above is given by

$$\theta^{ijk}(t) = \theta^{ijk} \frac{1}{1 - \frac{t}{6} (\theta^{a_1 a_2 a_3} H_{a_1 a_2 a_3})}. \quad (5.28)$$

5.3 An all order solution to the Seiberg-Witten map

An all order solution to the Seiberg-Witten map can be constructed using the flow discussed in the previous subsection. We first construct the Seiberg-Witten map for the fields b^i and \hat{b}^i . Our solution to the Seiberg-Witten map is as follows:

$$\rho(x^i) = x^i + \hat{b}^i = e^{\partial_t + \frac{1}{2} \theta^{ijk}(t) b_{ij} \partial_k} x^i \Big|_{t=0}. \quad (5.29)$$

We first check that (5.29) leads to the correct infinitesimal gauge transformations (5.3). It is also possible to write down the Seiberg-Witten map for finite gauge transformations, as opposed to infinitesimal gauge transformations considered here. We derive the Seiberg-Witten map for finite gauge transformation parameters in the appendix D.

From the map (5.29), the infinitesimal gauge transformation in the NP description induced by the infinitesimal gauge transformation of the ordinary description is given by

$$\begin{aligned} &\hat{b}^i(b + \delta_\Lambda b) - \hat{b}^i(b) \\ &= \left[e^{\partial_t + \frac{1}{2} \theta^{ijk}(t) (b_{ij} + \partial_i \Lambda_j - \partial_j \Lambda_i) \partial_k} - e^{\partial_t + \frac{1}{2} \theta^{ijk}(t) b_{ij} \partial_k} \right] x^i \Big|_{t=0}. \end{aligned} \quad (5.30)$$

Below we demonstrate that (5.30) indeed gives a solution to the Seiberg-Witten map, i.e. satisfies the condition (5.1). Recall that the gauge transformation laws for the fields b^i and \hat{b}^i are given in (5.3). Let us write

$$A \equiv \partial_t + \frac{1}{2} \theta^{ijk}(t) b_{ij} \partial_k, \quad B \equiv \frac{1}{2} \theta^{ijk}(t) (\partial_i \Lambda_j - \partial_j \Lambda_i) \partial_k, \quad (5.31)$$

Using this notation, (5.30) can be rewritten as

$$(e^{A+B} x^i - e^A x^i) \Big|_{t=0} = ([e^{A+B} e^{-A} - 1] e^A x^i) \Big|_{t=0}. \quad (5.32)$$

Eq. (5.32) involves the quantity

$$e^{A+B}e^{-A} = e^{h(A,B)}, \quad (5.33)$$

where the $h(A,B)$ is the linear combination of the terms derived by the Baker-Campbell-Hausdorf formula. In the case of infinitesimal gauge transformation, i.e. infinitesimal Λ , among the terms in $h(A,B)$ only the terms linear in B is relevant. Such term has the following form:

$$[\cdots [A, [A, B]] \cdots]. \quad (5.34)$$

Noting that the ∂_t term in A has a constant coefficient and there is no ∂_t term in B , (5.33) can be written as

$$e^{A+B}e^{-A} - 1 = \hat{\kappa}^i(t)\partial_i + \mathcal{O}(\Lambda^2). \quad (5.35)$$

Using (5.35), (5.30) can be written as

$$\hat{b}^i(b + \delta_\Lambda b) - \hat{b}^i(b) = \hat{\kappa}^i + \hat{\kappa}^j \partial_j \hat{b}^i, \quad (5.36)$$

where we identify the gauge transformation parameter in the NP description as

$$\hat{\kappa}^i \equiv \hat{\kappa}^i(t=0). \quad (5.37)$$

In the appendix C, we show that the gauge transformation parameter $\hat{\kappa}^i$ defined in (5.30) satisfies the divergenceless condition

$$\partial_i \hat{\kappa}^i = 0. \quad (5.38)$$

Therefore, it can be written as $\hat{\kappa}^i = \theta^{ijk} \partial_j \hat{\Lambda}_k$. Thus, \hat{b}^i given by (5.29) is a solution of the Seiberg-Witten map (5.1).

Next, let us construct the Seiberg-Witten map for fields φ and $\hat{\varphi}$. The gauge transformation laws for φ and $\hat{\varphi}$ are given in (5.7). A solution to the Seiberg-Witten map can be obtained as

$$\hat{\varphi} = e^A \varphi \Big|_{t=0}. \quad (5.39)$$

By a calculation similar to the case in \hat{b}^i , one can show that $\hat{\varphi}$ defined in (5.39) is a solution to the Seiberg-Witten map (5.1).

The gauge transformation law for $\hat{b}_{\mu i}$ is given in (5.6). To obtain a solution to the Seiberg-Witten map, it is useful to notice that $\hat{b}_{\mu i}$ appears in the covariant derivative (4.55). Therefore, we first look for a differential operator in the commutative description which (i) when acted on a scalar, the covariant derivative of the scalar transforms as a scalar under the volume-preserving diffeomorphisms; (ii) contains $b_{\mu i}$ linearly. Then, we consider an exponential map similar to

the Seiberg-Witten map for the scalar fields. In this way, we make the following guess for the solution to the Seiberg-Witten map:

$$\begin{aligned} & \overset{\circ}{\partial}_\mu - \hat{B}_\mu{}^i \overset{\circ}{\partial}_i \\ = & e^A \left(\overset{\circ}{\partial}_\mu - \theta^{ijk}(t) \left(\overset{\circ}{\partial}_j b_{\mu k} - \frac{1}{2} \partial_\mu b_{jk} \right) \overset{\circ}{\partial}_i \right) e^{-A} \Big|_{t=0}. \end{aligned} \quad (5.40)$$

where we use $\overset{\circ}{\partial}$ to denote differential *operators* emphasizing that the differential acts on the whole objects in the right, for example

$$\overset{\circ}{\partial}_t t = t \overset{\circ}{\partial}_t + 1, \quad \overset{\circ}{\partial}_i x^j = x^j \overset{\circ}{\partial}_i + \delta_i^j, \quad \overset{\circ}{\partial}_i f(x) = \partial_i f(x) + f(x) \overset{\circ}{\partial}_i. \quad (5.41)$$

(5.40) turns out to be indeed a solution to the Seiberg-Witten map. In appendix E, we show that $\hat{B}_\mu{}^i$ satisfies the divergenceless condition:

$$\partial_i \hat{B}_\mu{}^i = 0, \quad (5.42)$$

and hence can be written as

$$\hat{B}_\mu{}^i = \theta^{ijk} \partial_j \hat{b}_{\mu k}. \quad (5.43)$$

On the other hand, we can show the gauge transformation law for $\hat{B}_\mu{}^i$ in a way similar to the previous cases.

Our solution of the Seiberg-Witten map (5.29), (5.39) and (5.40) are expressed in a covariant tensor notation, and hence does not depend on particular coordinates. On the other hand, when one would like to study the small Nambu-Poisson tensor expansion, one should fix the open membrane metric as in (4.45) to compare the physical strength of the interaction through the Nambu-Poisson structure. Then we can more precisely specify the expansion as one in the small Nambu-Poisson tensor component in the y -coordinates (4.49), when we take the closed membrane metric as in (4.43).

5.4 Ambiguities in the SW map

The Seiberg-Witten map is not unique [64, 65]. There are two sources of ambiguities. The first one arises from the fact that the Seiberg-Witten map is basically a map between gauge orbits in two descriptions of the same theory. Therefore, replacing b_{ij} in the expression (5.29) by $b_{ij} + \delta_\Lambda b_{ij}$ for any gauge transformation Λ , for example, gives another Seiberg-Witten map, because a gauge transformation on b_{ij} does not affect the identification of the gauge orbit. Even if we insist that the Seiberg-Witten map should not involve anything other than b^i , θ^{ijk} , C_{ijk} and ∂_i , we can still get a new Seiberg-Witten map by taking $\Lambda_i = C_{ijk} b^j \partial_l b^k b^l$, for instance. This is the only ambiguity at the order of $\mathcal{O}(\theta^2)$ under such assumptions, and there is no ambiguity

of this kind at lower orders. But apparently at higher orders there are more and more such ambiguities. In general, the SW map for \hat{b}^i , for example, can be of the form

$$\hat{b}^i = \left(e^{\partial_t + \frac{1}{2}\theta^{ijk}(t)(b_{ij} + \partial_i \Lambda_j - \partial_j \Lambda_i)\partial_k} - 1 \right) x^i \Big|_{t=0} \quad (5.44)$$

for some vectors $\Lambda_i(b, \partial b, \dots)$ which are given functions of the dynamical fields.

The second source of ambiguities comes from field redefinitions. A simple way to see the potential existence of such ambiguities is the following. If we try to solve for the SW map order by order from the defining condition (5.1), we can always add a gauge invariant term with appropriate Lorentz transformation property at the n -th order in θ^{ijk} without spoiling the condition (5.1) at lower orders. Then we can solve for higher order terms of the SW map accordingly. The all order solutions of this type are of the form

$$\hat{b}^i = \left(e^{\partial_t + \frac{1}{2}\theta^{ijk}(t)b_{ij}\partial_k} - 1 \right) x^i \Big|_{t=0} + e^{\partial_t + \frac{1}{2}\theta^{ijk}(t)b_{ij}\partial_k} f^i(H, \partial H, \partial^2 H, \dots) \Big|_{t=0}, \quad (5.45)$$

where f^i is a local gauge invariant vector. As f^i is gauge invariant, such modifications of the SW map does not change the SW map for the gauge transformation parameter $\hat{\kappa}^i$ (5.35).

As an example, one can choose

$$f^i = \sum_{r=0}^{\infty} c_r G^r \theta^{ijk} \theta^{lmn} \partial_l H \partial_j \partial_m H \partial_k \partial_n H, \quad (5.46)$$

where c_r are arbitrary numbers and

$$H \equiv \theta^{ijk} H_{ijk}, \quad G \equiv \theta^{ijk} \theta^{lmn} \partial_i \partial_l H \partial_j \partial_m H \partial_k \partial_n H. \quad (5.47)$$

This example is constructed such that the coefficients c_r are all dimensionless and we have only used the minimal set of variables, i.e., the invariant tensor H_{ijk} and θ^{ijk} . Furthermore f^i is not divergenceless, so even at the lowest nontrivial order it can not be mistaken as the other kind of ambiguity due to a gauge transformation.

Since field redefinitions change the form of the action, this type of ambiguities may be fixed by restricting the form of the action [65]. Notice that our covariant tensor notation is useful for writing down all possible terms in the discussions of ambiguities.

Our solution to the Seiberg-Witten map has the clear meaning that it is a map between two different choices of coordinates, one which keeps the NP structure and the other so-called static gauge. It will be interesting to investigate further to what extent our solution is special among all possible solutions of Seiberg-Witten map.

6 Discussions

In this paper, we obtained several results which will be essential for showing the conjectured equivalence of the conventional M5-brane theory in a constant C -field background and NP M5-brane theory. The scaling limit discussed in section 3 is necessary for identifying the region of

validity of the NP M5-brane theory, and it also specifies how to take a limit in the conventional M5-brane theory in order to compare the two theories. The precise identification of the variables in the NP M5-brane theory with M-theory variables is another necessary step to relate the two M5-brane theories. The background-independent formulation was useful for reading off the open membrane metric and effective tension of the NP M5-brane. The all-order solution to the Seiberg-Witten map is of course essential for establishing the relation between two M5-brane theories. The logical steps we took to find the solution already indicates the equivalence of the two descriptions, as different choices of variables of the same theory. In the meantime, as pointed out in Ref.[2] the field $\hat{B}_\mu{}^i$ has similarity with the gauge field parameterizing complex structure deformations on a Calabi-Yau 3-manifold in the Kodaira-Spencer theory, and we feel our derivation has a room for mathematical sophistications.

Motivated by [2], a new covariant action for the self-dual 2-form gauge field which is based on the 3+3 decomposition of the worldvolume was formulated at the linear level in [27] (see [66] for a generalization to chiral p-form gauge field with general decomposition of the space-time). Since NP M5-brane theory is based on the 3+3 decomposition of the worldvolume, it seems more convenient to reformulate the conventional M5-brane theory also in the 3+3 decomposition of the worldvolume. Investigation in this direction will also be useful.

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A Absence of θ^{012} in the NP M5-theory scaling limit

We consider the scaling limit to the NP M5-brane theory (3.5):

$$\begin{aligned}\ell_P &\sim \epsilon^{1/3}, \\ g_{ij} &\sim \epsilon, \quad (i, j = 3, 4, 5) \\ C_{345} &\sim \epsilon^0.\end{aligned}\tag{A.1}$$

We define the (inverse) open membrane metric in (012)-directions in a similar way as in (4.27):

$$G_{OM}^{\mu\mu'} \equiv \frac{1}{2} \frac{\theta^{\mu\nu\rho}}{(2\pi)^2 \ell_P^3} \frac{\theta^{\mu'\nu'\rho'}}{(2\pi)^2 \ell_P^3} g_{\nu\nu'} g_{\rho\rho'} \quad (\mu, \nu, \dots = 0, 1, 2).\tag{A.2}$$

Here, as in (4.26), we postulate that the Nambu-Poisson tensor $\theta^{\mu\nu\rho}$ is given by

$$-C_{\mu\nu\rho}\theta^{\rho\beta\gamma} = \delta_\mu^\beta\delta_\nu^\gamma - \delta_\mu^\gamma\delta_\nu^\beta. \quad (\text{A.3})$$

As discussed in section 3, from the non-linear self-dual relation we obtain the scaling

$$C_{012} \sim \epsilon^{-1}, \quad (\text{A.4})$$

with the closed membrane metric scaling as

$$g_{\mu\nu} \sim \epsilon^0. \quad (\text{A.5})$$

From (A.3) and (A.4), we obtain

$$\theta^{012} \sim \epsilon, \quad (\text{A.6})$$

and thus θ^{012} vanishes in the scaling limit $\epsilon \rightarrow 0$. Notice that this is the result in which the (inverse) open membrane metric (A.2) is finite, thus it correctly measures the strength of the interaction through θ^{012} .

Let us study the compactification in x^2 direction to see, via the M-IIA relation, what the above scaling corresponds to in type IIA string theory. We take the scaling of the physical compactification radius R_2^{phys} as

$$R_2^{phys} \sim \epsilon^a. \quad (\text{A.7})$$

Then, from (A.5), the coordinate compactification radius R_2^{coord} also scales as

$$R_2^{coord} \sim \epsilon^a, \quad (\text{A.8})$$

where the number a is to be determined. The scalings of the type IIA variables are given as

$$\begin{aligned} \ell_s &= \left(\frac{\ell_P^3}{R_2^{phys}} \right)^{1/2} \sim \epsilon^{\frac{1-a}{2}}, \\ g_s &= \left(\frac{R_2^{phys}}{\ell_P} \right)^{3/2} \sim \epsilon^{\frac{3a-1}{2}}. \end{aligned} \quad (\text{A.9})$$

From (A.9) we should assume $a < 1$ to take the zero-slope limit in type IIA string theory. The B -field is related to the C -field as

$$B_{01} = C_{012}(R_2^{coord}) \sim \epsilon^{-1+a}. \quad (\text{A.10})$$

Then from (A.9)

$$2\pi\alpha' B_{01} \sim \epsilon^0. \quad (\text{A.11})$$

We will not take the OM-theory limit [45] (which is given by $(1 - 2\pi\alpha'B_{01})/\alpha' \sim \epsilon^0$). Then, the open string metric in (2.23) and the non-commutative parameter θ^{01} in (2.24) scale as

$$G_{\mu\nu} \sim \epsilon^0, \quad (\text{A.12})$$

$$\theta^{\mu\nu} \sim \alpha' \sim \epsilon^{1-a}, \quad (\mu, \nu = 0, 1). \quad (\text{A.13})$$

Thus in the scaling limit $\epsilon \rightarrow 0$, there is no interaction through the Poisson bracket in (01)-directions as long as $a < 1$.

As noted in subsection 4.2, our postulates for the open membrane metric and the Nambu-Poisson tensor reduces to the $\Phi = -B$ description (2.21) upon circle compactification. On the other hand, when $2\pi\alpha'B_{\mu\nu} \ll g_{\mu\nu}$, it may be better to use $\Phi = 0$ description in (2.20) instead of $\Phi = -B$ description above so that one can treat the open membrane metric as a small deformation from the closed string metric. The conclusion that $\theta^{01} \sim \epsilon^{1-a}$ does not change even in this case. On the other hand, we currently do not know how to uplift the freedom in the description in (2.20) to M-theory.

We may formally require the Yang-Mills coupling (2.28) on the D4-brane to be finite, although it is actually $U(1)$ gauge theory:

$$g_s \sim \epsilon^{\frac{3-p+r}{4}}. \quad (\text{A.14})$$

Here, $p = 4$ and r is the rank of the Poisson tensor finite in the scaling limit, which is zero as above. Comparing with (A.9), we have

$$a = \frac{1}{6}. \quad (\text{A.15})$$

On the other hand, in this compactification the interaction through the Nambu-Poisson bracket in (345)-directions is interpreted as induced on D4-brane by RR 3-form flux in type IIA string theory. Such a system has not been studied much previously. If we require finite interaction through the Nambu-Poisson bracket in (345)-directions in this theory, we obtain

$$a = 0. \quad (\text{A.16})$$

B (4.33) from the double dimensional reduction of NP M5-brane action

It has been shown in [2] that by the double dimensional reduction the NP M5-brane action reduces to the Poisson description of a D4-brane in a constant B -field background. Here, it will be enough to study the potential term of the NP M5-brane theory including X^5 given by (see (4.20) and (4.22)):

$$\frac{1}{(2\pi)^2 \ell_P^3} \int d^3x \int \frac{d^3x}{2\pi|\theta^{345}|} \frac{1}{(2\pi)^4 \ell_P^6} \frac{1}{12} g_{II'} g_{JJ'} g_{KK'} \{X^I, X^J, X^K\}, \{X^{I'}, X^{J'}, X^{K'}\}, \quad (I, J = 5, \dots, 10). \quad (\text{B.1})$$

The double dimensional reduction is described by taking

$$X^5 = x^5, \quad (\text{B.2})$$

where the coordinate compactification radius in the x^5 direction is R_5^{coord} . All the other fields are set to be independent of x^5 . Then, the potential term (B.1) becomes

$$\frac{1}{(2\pi)^2 \ell_P^3} \int d^3x \int \frac{d^2x (2\pi R_5^{coord})}{2\pi |\theta^{345}|} \frac{1}{(2\pi)^4 \ell_P^6} \frac{1}{4} g_{II'} g_{JJ'} g_{55} (\theta^{ij5} \partial_i X^I \partial_j X^J) (\theta^{i'j'5} \partial_{i'} X^{I'} \partial_{j'} X^{J'}), \quad (I, J = 6, \dots, 10). \quad (\text{B.3})$$

On the other hand, the potential term of the Poisson description of D4-brane is given by (see (4.1) and (4.11))

$$\frac{1}{(2\pi)^2 g_s \ell_s^3} \int d^3x \int \frac{d^2x}{2\pi |\theta^{34}|} \frac{1}{(2\pi \alpha')^2} \frac{1}{4} g_{II'} g_{JJ'} \{X^I, X^J\}_{Poisson} \{X^{I'}, X^{J'}\}_{Poisson}, \quad (I, J = 6, \dots, 10). \quad (\text{B.4})$$

where $\{*, *\}_{Poisson}$ is the Poisson bracket (4.13):

$$\{A, B\}_{Poisson} = \theta^{ij} \partial_i A \partial_j B, \quad (i, j = 3, 4). \quad (\text{B.5})$$

Comparing (B.3) and (B.4) using the M-IIA relation (2.14), we obtain

$$\theta^{34} = \frac{\theta^{345}}{2\pi R_5^{coord}}, \quad (\text{B.6})$$

which coincides with (4.33).

C Divergenceless condition for $\hat{\kappa}^i$

To make calculation simpler, we use the following explicit parametrization of θ^{ijk} :

$$\theta^{ijk} = \Theta \epsilon^{ijk}, \quad (\text{C.1})$$

where ϵ^{ijk} is a totally anti-symmetric tensor with $\epsilon^{345} = 1$. Similarly, we parametrize $\theta^{ijk}(t)$ in (5.27) as

$$\theta^{ijk}(t) = \Theta(t) \epsilon^{ijk}. \quad (\text{C.2})$$

In this parametrization, (5.27) takes the form

$$\partial_t \Theta(t) = \Theta^2(t) \partial \cdot b \quad (\text{C.3})$$

with $\Theta(0) = \Theta$.

A and B in (5.31) are now written as

$$A = \partial_t + \Theta(t)b^i\partial_i, \quad B = \Theta(t)\kappa^i\partial_i, \quad (\text{C.4})$$

where

$$b^i \equiv \frac{1}{2}\epsilon^{ijk}b_{jk}, \quad \kappa^i \equiv \epsilon^{ijk}\partial_j\Lambda_k. \quad (\text{C.5})$$

In this appendix we use notation for b and κ different from the main body (5.3) by the overall scaling. From the definition (C.5), κ^i satisfies the divergenceless condition

$$\partial_i\kappa^i = 0. \quad (\text{C.6})$$

Let us first calculate $[A, B]$:

$$\begin{aligned} & [A, B] \\ &= [\partial_t + \Theta(t)b \cdot \partial, \Theta(t)\kappa \cdot \partial] \\ &= \partial_t\Theta(t)\kappa \cdot \partial + \Theta(t)(b \cdot \partial\Theta(t))\kappa \cdot \partial + \Theta^2(t)b \cdot \partial\kappa \cdot \partial - \Theta(t)\kappa \cdot \partial(\Theta(t)b) \cdot \partial \\ &= \partial_t\Theta(t)\kappa \cdot \partial + \Theta(t)(\partial \cdot (b\Theta(t)) - (\partial \cdot b)\Theta(t))\kappa \cdot \partial \\ &\quad + \Theta(t)^2b \cdot \partial\kappa \cdot \partial - \Theta(t)\kappa \cdot \partial(\Theta(t)b) \cdot \partial \end{aligned} \quad (\text{C.7})$$

$$\begin{aligned} &= \Theta(t)\partial \cdot (b\Theta(t))\kappa \cdot \partial + \Theta^2(t)b \cdot \partial\kappa \cdot \partial - \Theta(t)\kappa \cdot \partial(\Theta(t)b) \cdot \partial \\ &= \Theta(t)\partial_i(\Theta(t)b^i\kappa^j - \Theta(t)b^j\kappa^i)\partial_j \equiv \Theta(t)\tilde{\kappa}_{(1)}(t) \cdot \partial, \end{aligned} \quad (\text{C.8})$$

where we have used (C.3) to go from (C.7) to the next line, and the divergenceless condition of κ to arrive at the last line. Notice that $\tilde{\kappa}_{(1)}$ in (C.8) satisfies the divergenceless condition:

$$\partial \cdot \tilde{\kappa}_{(1)} = \partial_i\partial_j(\Theta(t)b^{[i}\kappa^{j]}) = 0. \quad (\text{C.9})$$

Here, $[\]$ denotes the anti-symmetrization in indices.

Next, let us consider $[A, [A, B]]$. The calculation is almost the same to the above, expect that there is an explicit t -dependence in $\tilde{\kappa}_{(1)}(t)$. We obtain

$$[A, [A, B]] = \Theta(t) \left(\tilde{\kappa}'_{(2)}(t) + \partial_t\tilde{\kappa}_{(1)}(t) \right) \cdot \partial, \quad (\text{C.10})$$

where

$$\tilde{\kappa}'_{(2)}{}^j(t) \equiv \partial_i(\Theta(t)b^{[i}\tilde{\kappa}_{(1)}^{j]}(t)). \quad (\text{C.11})$$

However, the t -derivative on $\tilde{\kappa}_{(1)}^i(t)$ does not affect the divergenceless condition of $\tilde{\kappa}_{(1)}^i(t)$ (as long as $\tilde{\kappa}_{(1)}^i(t)$ is a smooth function of t and x). Thus, (C.10) can be again rewritten in a form

$$[A, [A, B]] = \Theta(t) \left(\tilde{\kappa}_{(2)}(t) \right) \cdot \partial, \quad (\text{C.12})$$

with divergenceless $\tilde{\kappa}_{(2)}^i(t)$.

Repeating the same arguments, since the $\Theta(0) = \Theta$ is constant, it becomes clear that $\hat{\kappa}(t)$ in (5.35) satisfies

$$\partial \cdot \hat{\kappa}(t) = 0. \quad (\text{C.13})$$

(C.13) is true for all t , including $t = 0$. Thus, $\hat{\kappa}^i = \hat{\kappa}^i(t = 0)$ is divergenceless when κ is divergenceless.

D Finite gauge transformations

Since the gauge symmetry in the commutative side is Abelian, the *finite* gauge transformation of b^i by a finite parameter Λ_i which we denote as b_Λ^i takes the same form as for the infinitesimal gauge transformations (5.3):

$$b_\Lambda^i = b^i + \theta^{ijk} \partial_j \Lambda_k. \quad (\text{D.1})$$

Therefore, the Seiberg-Witten map of the gauge transformed field b_Λ^i is again written in terms of B defined in (5.31) with finite Λ_i :

$$\hat{b}_\Lambda^i = (e^{A+B} - 1)x^i \Big|_{t=0}. \quad (\text{D.2})$$

This can be rewritten as

$$\hat{b}_\Lambda^i = \left(e^{A+B} e^{-A} \hat{b}^i(t) + (e^{A+B} e^{-A} - 1)x^i \right) \Big|_{t=0}, \quad (\text{D.3})$$

where

$$\hat{b}^i(t) \equiv (e^A - 1)x^i, \quad \hat{b}^i(t = 0) = \hat{b}^i. \quad (\text{D.4})$$

Using the Baker-Campbell-Hausdorff formula, (D.3) can be written as

$$\hat{b}_\Lambda^i = \left(e^{h(A,B)} \hat{b}^i(t) + (e^{h(A,B)} - 1)x^i \right) \Big|_{t=0}, \quad (\text{D.5})$$

where, as we will show below, $h(A, B)$ takes the form:

$$h(A, B) = \Theta(t) \hat{K}^i(t) \partial_i, \quad (\text{D.6})$$

where $\Theta(t)$ is defined in (C.2). Then, \hat{b}_Λ^i can be written as

$$\hat{b}_\Lambda^i = \left(e^{\Theta(t) \hat{K}^i(t) \partial_i} \hat{b}^i(t) + (e^{\Theta(t) \hat{K}^i(t) \partial_i} - 1)x^i \right) \Big|_{t=0}. \quad (\text{D.7})$$

As we will show shortly, $\hat{K}^i(t)$ satisfies the divergenceless condition

$$\partial_i \hat{K}^i(t) = 0. \quad (\text{D.8})$$

Therefore, $\hat{K}^i(t)$ can be written as

$$\hat{K}^i(t) = \epsilon^{ijk} \partial_j \hat{\Lambda}'_k(t). \quad (\text{D.9})$$

Taking $t = 0$ in (D.7), the finite gauge transformation of the field \hat{b}^i is given as

$$\hat{b}^i_{\Lambda} = \left(e^{\theta^{ljk}(\partial_j \hat{\Lambda}'_k) \partial_l} \hat{b}^i + \left(e^{\theta^{ljk} \partial_j \hat{\Lambda}'_k \partial_l} - 1 \right) x^i \right), \quad (\text{D.10})$$

where $\hat{\Lambda}'_k \equiv \hat{\Lambda}'_k(t=0)$, and θ^{ijk} and Θ^{ijk} are related as in (C.1). Notice that the reparametrization from $\rho(x^i) = x^i + \hat{b}^i$ to $\rho_{\Lambda}(x^i) = x^i + \hat{b}^i_{\Lambda}$:

$$\rho_{\Lambda}(x^i) = x^i + \hat{b}^i_{\Lambda} = e^{\theta^{ljk}(\partial_j \hat{\Lambda}'_k) \partial_l} (x^i + \hat{b}^i) = e^{\theta^{ljk}(\partial_j \hat{\Lambda}'_k) \partial_l} \rho(x^i), \quad (\text{D.11})$$

is nothing but the finite form of the volume-preserving diffeomorphism.

The form of $h(A, B)$ in (D.6) and the divergenceless condition for \hat{K}^i

From the Baker-Campbell-Hausdorff formula, $h(A, B)$ is a sum of the terms which have the form of multiple commutations with A or B , for example

$$[\cdots, [B, [A, [A, [B, \cdots [A, B] \cdots]]]] \cdots]. \quad (\text{D.12})$$

To calculate such terms, we use the notation in appendix C (see (C.1) \sim (C.5)). According to (C.8) and (C.12), if $Z = \Theta(t) Z^i(t) \partial_i$ satisfies the divergenceless condition $\partial_i Z^i(t) = 0$, the quantity $[A, Z]$ takes the form

$$[A, Z] = \Theta(t) \tilde{Z}^i(t) \partial_i, \quad (\text{D.13})$$

with $\partial_i \tilde{Z}^i(t) = 0$. Namely,

$$\partial_i Z^i(t) = 0 \Rightarrow \partial_i \tilde{Z}^i(t) = 0. \quad (\text{D.14})$$

Next, we show that if $Z = \Theta(t) Z^i(t) \partial_i$ satisfies the divergenceless condition $\partial_i Z^i(t) = 0$, $[B, Z]$ also takes the form

$$[B, Z] = \Theta(t) \tilde{Z}^i(t) \partial_i, \quad (\text{D.15})$$

with $\partial_i \tilde{Z}^i(t) = 0$. Namely,

$$\partial_i Z^i(t) = 0 \Rightarrow \partial_i \tilde{Z}^i(t) = 0. \quad (\text{D.16})$$

By a direct calculation, we have

$$\begin{aligned} [B, Z] &= [\Theta(t) \kappa^j \partial_j, \Theta(t) Z^i(t) \partial_i] \\ &= \Theta(t) (\kappa^j \partial_j \Theta(t) Z^i(t) + \Theta(t) \kappa^j \partial_j Z^i(t)) \partial_i - (\kappa \leftrightarrow Z(t)) \\ &\equiv \Theta(t) \tilde{Z}^i(t) \partial_i. \end{aligned} \quad (\text{D.17})$$

Here, we used B defined in (C.4), and κ^i is the one defined in (C.5) which is a different notation from the main body. From (D.17) we see that $[B, Z]$ has the form (D.15). On the other hand, the divergence of $\tilde{Z}^i(t)$ becomes

$$\begin{aligned} \partial_i \tilde{Z}^i(t) &= \partial_i \kappa^j \partial_j \Theta(t) Z^i(t) + \kappa^j Z^i(t) \partial_j \partial_i \Theta(t) + \partial_i \Theta(t) \kappa^j \partial_j Z^i(t) + \Theta(t) \partial_i \kappa^j \partial_j Z^i(t) \\ &\quad - (\kappa \leftrightarrow Z(t)). \end{aligned} \quad (\text{D.18})$$

Since the first line of (D.18) is symmetric under the exchange $\kappa^i \leftrightarrow Z^i(t)$, the right hand side of (D.18) vanishes:

$$\partial_i \tilde{Z}^i(t) = 0. \quad (\text{D.19})$$

Thus we have proved (D.16). From (D.13) and (D.15), $h(A, B)$ can be written in a form in (D.6):

$$h(A, B) = \Theta(t) \hat{K}^i(t) \partial_i, \quad (\text{D.20})$$

where from (D.14) and (D.16) $\hat{K}^i(t)$ is divergenceless:

$$\partial_i \hat{K}^i(t) = 0, \quad (\text{D.21})$$

which also holds for $t = 0$.

E Divergenceless condition for \hat{B}_μ^i

We first introduce a short hand notation for the adjoint action of A which we will use repeatedly:

$$(\mathbf{Ad}[A])C \equiv [A, C], \quad (\text{E.1})$$

for any operator C .

Let us consider the operator appearing in (5.40):

$$\hat{B}_\mu^i \partial_i \equiv \left(e^A (\overset{\circ}{\partial}_\mu - \Theta(t) (B_\mu^i - \partial_\mu b^i) \overset{\circ}{\partial}_i) e^{-A} - \overset{\circ}{\partial}_\mu \right) \Big|_{t=0}, \quad (\text{E.2})$$

where $B_\mu^i \equiv \epsilon^{ijk} \partial_j b_{\mu k}$. As in the appendix C, in this appendix we use notation for B_μ^i different from that in the main body (5.12) by the overall scaling. We can decompose the operator in the right hand side of (E.2) (before taking $t = 0$) into three parts:

- I. The term arising from the exponential map of $\overset{\circ}{\partial}_\mu$.
- II. The term arising from the exponential map of $\Theta(t) B_\mu^i$.
- III. The term arising from the exponential map of $\Theta(t) \partial_\mu b^i$.

As discussed in the previous section, the term [II] satisfies the divergenceless condition by itself since $\partial_i B_\mu^i = 0$. Therefore, the remaining thing to show is that [I]+[III] satisfies the divergenceless condition.

Let us first look at [I] (here we also include the term $-\partial_\mu$ in the right hand side of (E.2) in this category):

$$\begin{aligned}
e^A \overset{\circ}{\partial}_\mu e^{-A} - \overset{\circ}{\partial}_\mu &= \sum_{n=0}^{\infty} \frac{1}{n!} (\mathbf{Ad}[A])^n \overset{\circ}{\partial}_\mu - \overset{\circ}{\partial}_\mu \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} (\mathbf{Ad}[A])^{n-1} (-\partial_\mu (\Theta(t) b^i \partial_i)) \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\mathbf{Ad}[A])^n (-t\Theta(t)(\Theta(t)\partial_\mu(\partial \cdot b)b^i \partial_i) - \Theta(t)(\partial_\mu b^i \partial_i)) \quad (\text{E.3})
\end{aligned}$$

where we have used

$$\partial_\mu \Theta(t) = t\Theta^2(t)\partial_\mu(\partial_i b^i), \quad (\text{E.4})$$

which follows directly from (C.3). As a short hand notation, we define

$$\alpha_\mu \equiv (\Theta(t)\partial_\mu(\partial \cdot b)b^i \partial_i) \equiv \alpha_\mu^i \partial_i, \quad (\text{E.5})$$

$$\beta_\mu \equiv (\partial_\mu b^i \partial_i) \equiv \beta_\mu^i \partial_i. \quad (\text{E.6})$$

Then, (E.3) can be written as

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\mathbf{Ad}[A])^n (-t\Theta(t)(\Theta(t)\partial_\mu(\partial \cdot b)b^i \partial_i) - \Theta(t)(\partial_\mu b^i \partial_i)) \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\mathbf{Ad}[A])^n (-t\Theta(t)\alpha_\mu - \Theta(t)\beta_\mu) \\
&= -t\Theta(t)\alpha_\mu - \Theta(t)\beta_\mu \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} (-t(\mathbf{Ad}[A])^n (\Theta(t)\alpha_\mu) - n(\mathbf{Ad}[A])^{n-1} (\Theta(t)\alpha_\mu) - (\mathbf{Ad}[A])^n (\Theta(t)\beta_\mu)).
\end{aligned} \quad (\text{E.7})$$

When we take $t = 0$, the relevant part of the operator is given by

$$e^A \overset{\circ}{\partial}_\mu e^{-A} - \overset{\circ}{\partial}_\mu = -\Theta(t)\beta_\mu + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} (-n(\mathbf{Ad}[A])^{n-1} (\Theta(t)\alpha_\mu) - (\mathbf{Ad}[A])^n (\Theta(t)\beta_\mu)). \quad (\text{E.8})$$

Next, we turn to the term [III] which is given by

$$e^A (\Theta(t)\partial_\mu b^i \overset{\circ}{\partial}_i) e^{-A} = \Theta(t)\partial_\mu b^i \partial_i + \sum_{n=1}^{\infty} \frac{1}{n!} (\mathbf{Ad}[A])^n (\Theta(t)\partial_\mu b^i \partial_i). \quad (\text{E.9})$$

Using β_μ in (E.6), it is written as

$$e^A(\Theta(t)\partial_\mu b^i \overset{\circ}{\partial}_i)e^{-A} = \Theta(t)\beta_\mu + \sum_{n=1}^{\infty} \frac{1}{n!}(\mathbf{Ad}[A])^n(\Theta(t)\beta_\mu). \quad (\text{E.10})$$

Now we show that the sum of (E.8) and (E.10) is divergenceless. It is given by

$$+ \sum_{n=1}^{\infty} \left(\frac{1}{n!}(\mathbf{Ad}[A])^n(\Theta(t)\beta_\mu) - \frac{1}{(n+1)!} \left(n(\mathbf{Ad}[A])^{n-1}(\Theta(t)\alpha_\mu) + (\mathbf{Ad}[A])^n(\Theta(t)\beta_\mu) \right) \right), \quad (\text{E.11})$$

which becomes

$$- \sum_{n=1}^{\infty} \left(\frac{n}{(n+1)!}(\mathbf{Ad}[A])^{n-1}((\Theta(t)\alpha_\mu) - (\mathbf{Ad}[A])(\Theta(t)\beta_\mu)) \right). \quad (\text{E.12})$$

Thus we need to show that γ_μ^i defined by

$$\Theta(t)\gamma_\mu^i \partial_i \equiv (\Theta(t)\alpha_\mu) - (\mathbf{Ad}[A])(\Theta(t)\beta_\mu), \quad (\text{E.13})$$

satisfies the divergenceless condition:

$$\partial_i \gamma_\mu^i = 0. \quad (\text{E.14})$$

This can be shown by a direct calculation.

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